P177 见 Lemma 10.6 + prop 10.6

prop 10.6. $\mathfrak{g}(A)$ $L(\Lambda)$ $\Lambda \in P_+$ Then

(1) $Y(L(\Lambda))$ is solid convex W-invariant set. for every $x \in \operatorname{Int} X_C$

$tx \in Y(L(\Lambda))$ for $t \in \mathbb{R}_+$ sufficiently large

(2) $\operatorname{ch} L(\Lambda)$ is a holomorphic function on $\operatorname{Int} Y(L(\Lambda))$

(3) $Y(L(\Lambda)) \supset \operatorname{Int} Y$ A inde. sy $Y(L(\Lambda)) \subset \operatorname{Int}(Y)$ Y is open

(4) $\sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)}$ converges absolutely on $\operatorname{Int} X_C$ to a holomorphic function and diverges absolutely on $\mathfrak{h} \setminus \operatorname{Int} X_C$

is A - sym.. $\operatorname{ch} L(\Lambda)$ can be extend from $Y(L(\Lambda)) \cap X_C$ to a mermorphic function on $\operatorname{Int} X_C$

✓ A → indecomposable sy $Y(L(\Lambda)) = Y$

Lemma 10.6. set $Y = \{ h \in \mathfrak{g} \mid \sum_{\alpha \in \Delta_+} (\operatorname{mult} \alpha) |e^{-\alpha(h)}| < \infty \}$

$Y_N = \{ h \in \mathfrak{g} \mid \operatorname{Re}(\alpha_i(h)) > N \}$ $i = 1 \sim n$ for $N \in \mathbb{R}_+$

$Y \subset X_C$ $X_C = \bigcup_{w \in W} w(Y_0)$

(1) (a) V h.w.m over $\mathfrak{g}(A)$

(b) $Y(V)$ is a convex set

(2) $Y(V) \supset Y \cap Y_0$

(3) $Y(V) \supset Y_{\log n}$

§ 11.9

Fix $\alpha \in \Delta_+$ and set

$$n_{\pm}^{(\alpha)} = \bigoplus_{j=1}^{+\infty} \mathfrak{g}_{\pm j\alpha} \qquad \mathfrak{g}^{(\alpha)} = n_{\pm}^{(\alpha)} + \mathbb{C}\nu^{-1}(\alpha)$$

claim: $\mathfrak{g}^{(\alpha)}$ is a subalgebra of $\mathfrak{g}(A)$ By Thm 2.2 C

pf $[x, y] = (x|y)\, \nu^{-1}(\alpha)$ for $x \in \mathfrak{g}_\alpha$ $y \in \mathfrak{g}_{-\alpha}$

$\lambda \in P_+$

- $\alpha \in \Delta_+^{re}$, $g^{(\alpha)} \cong sl_2(\mathbb{C})$, module $L(\Lambda)$ restricted to $g^{(\alpha)}$ decomposes into a direct sum of irreducible f.d. - module (prop 3.6(a) integrable)
- $\alpha \in \Delta_+^{im}$, $g^{(\alpha)}$ is an infinite-dimensional Lie alge. (Cor 9.12 $\mathbb{C}\alpha^\vee(\alpha) \oplus \bigoplus_{j\neq 0} g_{j\alpha}$ is an infinite Heisenberg Lie alge.

aim: we describe the restriction of $L(\Lambda)$ to $g^{(\alpha)}$ for $\alpha \in \Delta_+^{im}$

prop 11.9. Let $\alpha \in \Delta_+^{im}$, and $\Lambda \in P_+$, two subspace of $L(\Lambda)$

$$\left.\begin{aligned} L(\Lambda)_0^{(\alpha)} &:= \bigoplus_{\lambda : (\lambda|\alpha) = 0} L(\Lambda)_\lambda \\ L(\Lambda)_+^{(\alpha)} &:= \bigoplus_{\lambda : (\lambda|\alpha) > 0} L(\Lambda)_\lambda \end{aligned}\right\}$$

(a) as $g^{(\alpha)}$ - module, $L(\Lambda)$ decomposes into a direct sum of submodule $L(\Lambda) = L(\Lambda)_0^{(\alpha)} \oplus L(\Lambda)_+^{(\alpha)}$

(b) $L(\Lambda)_0^{(\alpha)} = \{ x \in L(\Lambda) \mid g^{(\alpha)}(x) = 0 \}$

(c) $L(\Lambda)_+^{(\alpha)}$ is a free $U(n_-^{(\alpha)})$ - module on a basis of subspace $\{ x \in L(\Lambda)_+^{(\alpha)} \mid n_+^{(\alpha)}(x) = 0 \}$

(d) The $g^{(\alpha)}$ module $L(\Lambda)$ is completely reducible.

Pf: for (b), 9.12(b), $L \subset \Delta_+$ s.t

(i) $(\alpha|\beta) \neq 0$ real number of the same sign, for all $\alpha, \beta \in L$

(ii) $\alpha, \beta \in L$ and $\alpha - \beta \in \Delta_+ \Rightarrow \alpha - \beta \in L$.

the $g^L = n_+^L \oplus \mathfrak{h}^L$ is isomorphic to quotient $g'(B)$

$L(\Lambda)_0^{(\alpha)} = \{x \in L(\lambda) \mid \mathfrak{g}^{(\alpha)}(x) = 0\}$

[prop 11.8 $\mathfrak{a} \subset \mathfrak{g}(A)$ w_0-invariant subalgebra and for an $h \in \text{Int}\, X$ $[h, \mathfrak{a}] = \mathfrak{a}$,

$L(\Lambda)$ $(\Lambda \in P_+)$ decomposable into an orthogonal direct sum of irreducible h-invariant submodules]

Let $\mathfrak{a} = \mathfrak{g}^{(\alpha)}$ $(\alpha \in \Delta^+_{im})$

$\mathfrak{g}^{(\alpha)} = n_-^{(\alpha)} \oplus \mathbb{C}\nu^{-1}(\alpha) \oplus n_+^{(\alpha)}$

w_0 ✓

$\nu^{-1}(\alpha) = h$ $\quad L(\lambda)$

Each of these submodules is clearly generated By a nonzero vector $v_\lambda \in L(\lambda)_\lambda$ s.t. $n_+^{(\alpha)}(v_\lambda) = 0$

$n_-^{(\alpha)}(v_\lambda) = 0$

$\mathbb{C}\nu^{-1}(\alpha)(v_\lambda) = 0$

Let $v \in L(\lambda)_0^{(\alpha)} \Rightarrow \mathfrak{g}^{(\alpha)}(v) = 0$

$\quad \| $

$\bigoplus_{\lambda : (\lambda|\alpha) = 0} L(\lambda)_\lambda$

$v = v_\lambda \in L(\lambda)_\lambda$ and $(\lambda|\alpha) = 0 \Rightarrow n_-^{(\alpha)}(v_\lambda) = 0$

$\forall x \in n_-^{(\alpha)}$ $\quad n_+^{(\alpha)}(v_\lambda)$

$H(x(v_\lambda), x(v_\lambda)) = 0$

$= -H(v_\lambda, (w_0(x) \cdot x)(v_\lambda))$

$x \in n_-^{(\alpha)}$ $\quad w_0(x) \in n_+^{(\alpha)}$

$(w_0(x) \cdot x) \cdot (v_\lambda) = [w_0(x), x](v_\lambda)$

$= (w_0(x) | x)\, \nu^{-1}(\alpha)(v_\lambda) = (w_0(x)|x)(\lambda|\alpha) \cdot v_\lambda$

$\qquad (\lambda|\alpha) = 0$

$e_{\nu^{-1}(\alpha)} \cdot v_\lambda = \mathbb{C}\,(\lambda|\alpha)\, v_\lambda = 0 \Rightarrow g^{(\alpha)}(v_\lambda) = 0$

$\forall\, v \in L(\lambda)_0^{(\alpha)}, \quad g^{(\alpha)}(v) = 0$

" $\subseteq$ "

" $\supseteq$ " $\{x \in L(\lambda) \mid g^{(\alpha)}(x) = 0\}$

$n_-^{(\alpha)}(v) = 0 \Rightarrow (\lambda|\alpha) = 0$

$\dim(L(\lambda)) = 1 \Leftrightarrow \lambda|_{\mathfrak{h}'} = 0$

$g^{(\alpha)} \longrightarrow \dim(\mathbb{C}(v_\lambda)) = 1$; $L(\lambda)_0^{(\alpha)} \leftarrow (\lambda|\alpha) = 0 \nearrow \lambda(\nu^{-1}(\alpha)) = (\lambda|\alpha) = 0$

(c) for $v_\lambda \in L(\lambda)_+^{(\alpha)}$, $(\lambda|\alpha) > 0$

recall prop 9.10 (a) $M(\lambda)$ is irreducible if $2\langle\lambda+\rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta)$ for $\beta \in Q_+ \setminus \{0\}$, the $g'(A)$ - module

the $U(n_-^{(\alpha)})(v_\lambda)$ is the quotient of the corresponding Verma $g^{(\alpha)}$ - module $M(\lambda)$

$(\lambda|\alpha) > 0 \Rightarrow \langle\lambda+\rho, 2\nu^{-1}(\alpha)\rangle \neq (\alpha|\alpha)$

$(\rho|\alpha) > 0$

$(\alpha|\alpha) \leq 0$

$U(n_{-1}^{(\alpha)})(v_\lambda) = M(\lambda)$

$[U(n_{-1}^{(\alpha)})(v_\lambda)]_0 = \mathbb{C} v_\lambda$

$$[U(n_{-1}^{(\alpha)})(v_\lambda)]_k = \sum_{\substack{i_1, \dots, i_s \in \mathbb{Z}_+ \\ i_1 + \dots + i_s = k}} g_{-i_1\alpha} \cdots g_{i_s\alpha}(v_\lambda)$$

Then $[U(n_{-1}^{(\alpha)})(v_\lambda)]_j \subset L(\lambda)_{(\lambda - j\alpha)}$

$(\lambda - j\alpha | \alpha) > 0$

$L(\Lambda)_+^{(\alpha)}$ $(\lambda|\alpha) > 0$

$\forall$ submodule $\{ x \neq 0 \in L(\Lambda)_\lambda \}$ $n_+^{(\alpha)}(x) = 0$

$(\lambda|\alpha) \geq 0$ $\forall \lambda \in P(L(\Lambda))$

Cor 10.1 If $\Lambda \in P_+ \Rightarrow \lambda \in P(L(\Lambda))$ $\lambda \xrightarrow{w} u \in P_+ \cap P(L(\Lambda))$

$(\lambda|\alpha) = (w(\lambda) | w(\alpha)) \geq 0$

$\hat{P}_+$ $\in \Delta_+^{im}$

Cor 11.9 $\alpha \in \Delta_+^{im}$, $\Lambda \in P_+$, $\lambda \in P(L(\Lambda))$, Then either

(a) $(\lambda|\alpha) = 0$, then $\lambda - k\alpha \notin P(L(\Lambda))$ for $k \neq 0$ or else

$\{x | g^{\alpha}(x) = 0\}$ $n_-^{(\alpha)}$ $n_+^{(\alpha)}$

(b) $(\lambda|\alpha) \neq 0$, $(\lambda|\alpha) > 0$: (虚根的权链)

(i) $t \in \mathbb{Z}$ s.t $\lambda - t\alpha \in P(L(\Lambda))$

$t \to [-p, +\infty]$, Where $p \geq 0$

$t \mapsto \text{mult}_{L(\lambda)}(\lambda - t\alpha)$ is a nondecreasing function

(ii) $x \in g_{-\alpha}$ $x \neq 0$, $x: L(\Lambda)_{\lambda - t\alpha} \to L(\Lambda)_{\lambda - (t+1)\alpha}$ is an injective

$\dim L(\Lambda)_{\lambda - t\alpha} < \dim L(\Lambda)_{(\lambda - (t+1)\alpha)}$

(c) If $\lambda \in P_{++}$, $v \leftarrow \lambda$

$n_- \to L(\Lambda)$ defined by $n \mapsto n(v)$ is injective.

$n \neq 0$, $n(v) \neq 0$ $n \in n_-$ $\Leftarrow$ $n(v) = 0 \Rightarrow n = 0$

$v \neq 0$

pf: fix $0 \neq x$: $g_{-\alpha}$ $L(\Lambda) \hookrightarrow L(\Lambda)$ is an injective

Choose $y \in g_\alpha$. s.t $[x, y] = k = \sum_{i=1}^{n} a_i^\vee \alpha_i$

$= \nu^{-1}(S)$

let $v \in L(\Lambda)$ and $v \neq 0$

s.t $x(u) = 0$ $\Leftarrow$ $xy^n(u) = \Lambda(k) n y^{n-1}(u)$

$x(y(u)) = [x,y]u + y\,x(u) = [x,y](u) = \Lambda(k)\cdot u$, >0, $\neq 0$

$\neq 0$

$(\lambda|\alpha) > 0$

$\lambda(\alpha_i^\vee) > 0$

$\Lambda(\alpha_i^\vee) > 0$

$\Lambda(k) > 0$

$xy^n(u) = [x,y]y^{n-1}(u) + y(xy^{n-1}(u))$

$= k\cdot(y^{n-1}(u)) + y(xy^{n-1}(u))$

$= \Lambda(k)\cdot y^{n-1}(u) + y(\Lambda(k)(n-1)y^{n-2}(u))$

$= (\Lambda(k) + (\Lambda(k))(n-1))y^{n-1}(u)$

$= \Lambda(k)\,n\,y^{n-1}(u)$

$v \neq 0 \Rightarrow y(v) \neq 0 \Rightarrow y^2(v) \cdots \quad y^n(v) \neq 0$

$v = 0$

§ 11.10 (destribe explicity the region of convegence of $\mathrm{ch}\,L(\lambda)$

prop: 11.10 A be an indecomposable GCM. $L(\Lambda)$, $\Lambda \in P_+$ s.t $\langle \Lambda, \alpha_i^\vee \rangle \neq 0$ for some i

$\Lambda(k) > 0$

Then $Y(L(\Lambda)) \subset Y \Leftarrow \mathrm{ch}_{L(\lambda)} = \sum_{\lambda \in P(L(\Lambda))} \mathrm{mult}(\lambda)\, e^{\lambda(h)}$

$\parallel$ $\Uparrow$

$\Vert$ $\Downarrow$ coincides with the set

$$Y = \{ h \in \mathfrak{h} \mid \sum_{\alpha\in\Delta_+} (\operatorname{mult}\alpha)\, |e^{-\alpha(h)}| < \infty \}$$

Pf: ① $Y(L(\Lambda)) \subset Y$

② Y is open ($Y(L(\Lambda)) \supset \operatorname{Int} Y$)

For ① A is of finit type.

$$ch_{L(\Lambda)} = \sum_{\lambda\in P(\Lambda)} e^{\lambda(h)} < \infty$$

$$\geq \sum_{\alpha\in\Delta_+} |e^{(\Lambda-\alpha)(h)}| = e^{\Lambda(h)} \sum_{\alpha\in\Delta_+} e^{-\alpha(h)} < \infty$$

$$\sum_{\alpha\in\Delta_+} |e^{-\alpha(h)}| < \infty$$

$$Y = \mathfrak{h} = Y(L(\Lambda))$$

② A is of affine type.

$$\{ h\in\mathfrak{h} \mid \operatorname{Re}\delta(h) > 0 \} = Y = \{ h\in\mathfrak{h} \mid \sum_{\alpha\in\Delta_+} \operatorname{mult}(\alpha)\, |e^{-\alpha(h)}| < \infty \}$$

$\cup$

$Y(L(\Lambda))$

By 6.3 $\longrightarrow$ 仿射型实根. $\alpha + n\delta$, $\alpha \in \mathring{\mathfrak{g}}$

Cor 7.4 + Cor 8.3 (the mult (虚根) are bounded by M)

1. nontwisted affine Lie alg. of rank $l+1$

   $\operatorname{mult}(\text{imag... root}) = l$

2. X_N^r $\quad \operatorname{mult}(jr\delta) = l$

   $\operatorname{mult}(s\delta) = \left(\frac{N-l}{r-l} \right)$

$\exists M$ s.t mult (虚根) $< M$.

$Y = \{h \in \mathfrak{h} \mid \sum_{\alpha \in \Delta_+} \text{mult}\,\alpha \, |e^{-\alpha(h)}| < \infty\} \supseteq \{h \in \mathfrak{h} \mid \text{Re}\,\delta(h) > 0\}$

$Y(L(\Lambda)) \subset \{h \in \mathfrak{h} \mid \text{Re}\,\delta(h) > 0\}$ U1

$\text{ch}_{L(\Lambda)} = \sum_{\lambda \in P(L(\Lambda))} \text{mult}_{L(\Lambda)}(\lambda)\, e^{\lambda(h)} < \infty$ $\quad \delta(h) > 0$

$\alpha \in \Delta_+^{im}$

$t \longmapsto \text{mult}(\Lambda - s\alpha)$ 非减

$\lambda = \Lambda - s\delta \quad s \in \mathbb{Z}_+$

$\text{mult}(\Lambda - \delta) \neq 0 \quad \Rightarrow \text{Re}\,\delta(h) > 0$

$\Rightarrow \cdots \text{mult}(\Lambda - s\delta) \neq 0$

$\Lambda(\alpha_i^\vee) > 0$

3. A is indefinite type. A is indecomposable GCM.

By Thm 5.6 (c) (虚根存性定理)

$\exists\, \alpha \in \Delta_+^{im}$ s.t $\text{supp}\,\alpha = S(A)$ and $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all i

then $\lambda - \alpha \in P(\lambda)$ ✓

By Corollary. $\Lambda \in P_+$ $\Lambda(\alpha_i^\vee) \geq 0$. $\exists i$ s.t $\Lambda(\alpha_i^\vee) > 0$

$(\Lambda|\alpha) = \langle \Lambda, \sum k_i \nu^{-1}(\alpha_i) \rangle > 0$, $k_i > 0$, $\nu^{-1}(\alpha_i) = \alpha_i^\vee$

$\Rightarrow \Lambda - t\alpha \in P(\Lambda)$

where $t \in [-p, +\infty)$ $(p \geq 0)$

$t = 1$ $\qquad \Lambda - \alpha \in P(\Lambda)$ ✓

moreover By 11.1(b) + (Cor 11.9c.) for every nonzero $v \in L(\Lambda)_{\Lambda-\alpha}$ $(\Lambda-\alpha \in P(\Lambda))$

$\Lambda-\alpha \in P_+$ $(\Lambda-\alpha)(\alpha_i^\vee) = \Lambda(\alpha_i^\vee) - \alpha(\alpha_i^\vee) > 0$, $\alpha(\alpha_i^\vee) < 0$

the map $\psi : n_- \longrightarrow L(\Lambda)$ defined by

$\psi(y) = y(v)$ $y \longmapsto y(v)$, $\Lambda - \alpha$

$\Rightarrow \psi$ is injective.

$\lambda = \Lambda - \alpha$, $\mathrm{mult}_{L(\Lambda)}(\lambda) \neq 0$

$n_-(v) \neq 0$

$v \neq 0 \Rightarrow \alpha$ 取遍 Δ_+

$\Lambda - \alpha - s\alpha$

$Y(L(\Lambda)) \subset Y = \{ h \in \mathfrak{h} \mid \sum_{\alpha \in \Delta_+} \mathrm{mult}(\alpha) |e^{-\alpha(h)}| < \infty \}$

$\|$

绝对收敛区域

$$\sum_{\lambda \in P(\Lambda)} \mathrm{mult}(\lambda) e^{\lambda(h)} = \sum_{\alpha \in \Delta_+} (e^{\Lambda(h)}) \mathrm{mult}(\lambda) e^{-\alpha(h)}$$

$\Rightarrow Y$ is open

Dirichlet series

令 $s = \sigma + it$ σ, t

suppose $\sigma > \sigma_a$ is the absolute convergence region of $\sum_{n=1}^{\infty} a_n e^{-\lambda_n(z)}$ (or $\sum_{n=1}^{\infty} f(n) n^{-s}$)

Thm: 令 $f(s)$ 在半平面 $\sigma > c$ 的 Dirichlet series 表示为 $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$，其中 c 有限.

$a_n \geq 0$ for all $n \geq n_0$，若 $f(s)$ 在 $Re(s) > \sigma_0$ 上收敛. 且 f 在 $s = \sigma_0$ 的邻域上可以延拓成一个全纯函数，the series defining $f(s)$ converges for $Re(s) > \sigma_0 - \varepsilon$ for some $\varepsilon > 0$

([Ref: J-P-Serre chapter VI prop 7]
T. M. 阿普斯托尔：Introduction to Analytic number Theory)